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Note

A note on semiextensions of stable circuits

E. Enrique García Moreno [a], Tommy R. Jensen [b,*]

[a] CINVESTAV, Mexico City & Randolph College, Lynchburg, VA, Mexico
[b] Alpen-Adria-Universität Klagenfurt, Austria

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ABSTRACT

A semiextension of a circuit C in a graph G provides a possible means of finding a cycle double cover of G with C as a prescribed circuit. Recently we conjectured [E.E. García Moreno, T.R. Jensen, On semiextensions and circuit double covers, J. Combin. Theory Ser. B 97 (2007) 474–482] that if G is cubic and 2-edge-connected, then a semiextension of C in G exists. If true, this would imply several long-standing conjectures.

If there is a circuit C' in G with $C' \neq C$ and $V(C) \subseteq V(C')$, then C' is called an extension of C, a special case of a semiextension. If there is no such circuit, then C is said to be stable in G. Hence the existence question for semiextensions is easy except for stable circuits. Not many examples of graphs with stable circuits have been published. In this note we show that the members of a particular class of stable circuits described by M. Kochol have semiextensions.

1. Introduction

Let G be a graph with a circuit C. An *extension* of C is a circuit C' with $C' \neq C$ and $V(C) \subseteq V(C')$. If no extension of C exists in G, then C is called *stable* in G. A *semiextension in* G of C is a circuit D, $D \neq C$, such that for every path P in $G - (E(C) \cup E(D))$ from a vertex $x \in V(C) \setminus V(D)$ to a vertex $y \in V(C) \cup V(D)$ and having no interior vertices in $V(C) \cup V(D)$, there exists a path from x to y in G each of whose edges belongs to precisely one of C and D. Hence an extension is a special case of a semiextension.

Conjecture 1 (*[3]*). *If G is cubic and 2-edge-connected, and C is any circuit in G, then there exists a semiextension of C in G.*

It was shown in [3] that the truth of Conjecture 1 would imply that every 2-edge-connected graph allows a circuit double cover in which any one circuit has been prescribed. Fleischner [1] pointed out that there exist cubic 3-connected graphs with stable circuits; thus this strong version of the circuit double-cover conjecture cannot be shown only by means of finding extensions of circuits.

As an illustration of these concepts, Fig. 1 shows an example of a cubic graph H of order 20 with a circuit C (bold), for which a semiextension D is indicated (dashed). A tedious case analysis will show that C is stable in H (a fact which is however not too important for the purpose of the illustration). To confirm that D is a semiextension, one needs to investigate the paths v_3v_7, $v_{10}v_{15}$, $v_{11}v_{20}v_{14}$, $v_{11}v_{20}v_{18}$, and $v_{14}v_{20}v_{18}$, which are precisely the paths in $H - E(C \cup D)$ that join vertices of $C - V(D)$ to vertices of $C \cup D$. Since the symmetric difference of $E(C)$ and $E(D)$ induces a single circuit that contains all vertices of H except v_5 and v_{20}, and hence contains all endvertices of these paths, it follows that D satisfies the conditions for being a semiextension in H of C.

Not many examples of graphs with stable circuits have been published. In this note we show that the stable circuits in a class of snarks described in [5] allow semiextensions.

* Corresponding author.
E-mail address: tjensen@uni-klu.ac.at (T.R. Jensen).

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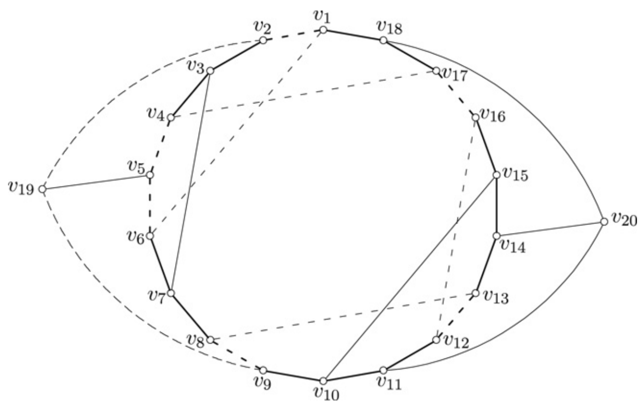

Fig. 1. The graph H with stable circuit (bold edges) and semiextension (dashed edges).

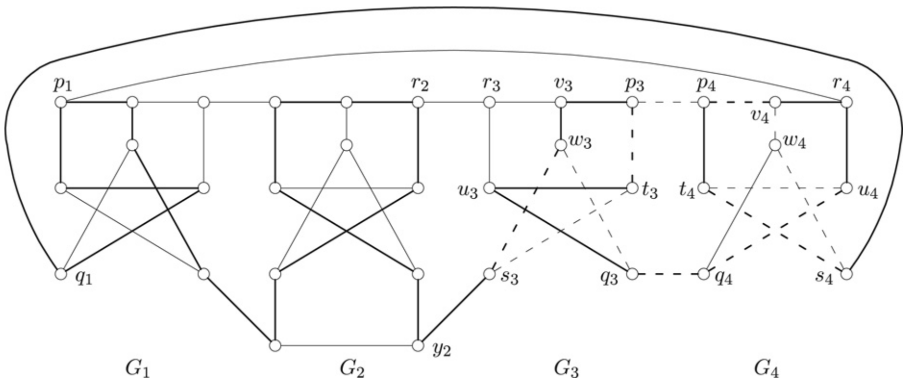

Fig. 2. The graph G with stable circuit C (bold edges) and circuit D (dashed edges).

2. Semiextensions of stable circuits in snarks

For a natural number k a graph G is *cyclically k-edge-connected* if the deletion of any set of fewer than k edges from G does not leave a graph with two components that both contain circuits. A *snark* is a cyclically 4-edge-connected cubic graph of girth at least 5 having no proper edge-coloring using three colors. It is not hard to see that if a cubic graph G allows a proper 3-edge-coloring, then any collection of disjoint circuits $C_1, C_2, \ldots, C_k$ may be extended to a circuit double cover of G. Applying elementary reductions of small edge cuts, it follows that if every stable circuit in an arbitrary snark has a semiextension, then the circuit double-cover conjecture is true.

Kochol [5] has exhibited infinite classes of snarks with stable circuits. For each of these graphs the possibility exists that it could be a smallest counterexample to the strong version of the circuit double-cover conjecture, as it allows neither a direct construction of a circuit double cover from a 3-edge-coloring, nor a reduction of the problem of extending its stable circuit C to a circuit double cover by using an extension of C. By demonstrating the existence of a semiextension of C, we rule out this possibility in each of the cases, and as it turns out, we may do so in a rather straightforward manner.

In Fig. 2 we reproduce, from [5], a cyclically 4-edge-connected cubic graph G and a circuit C in G. It is shown in [5] that G is a snark and that C is stable in G.

The vertices of G are partitioned into four subsets inducing subgraphs G_1, G_2, G_3, G_4 as indicated in Fig. 2. Using the notation as indicated, the circuit C is described by

$$C = s_3w_3v_3p_3t_3u_3q_3q_4u_4r_4v_4p_4t_4s_4Qs_3,$$

where Q is the subpath of C whose interior vertices lie outside the union of G_3 and G_4. Another circuit D shown in the figure is given by

$$D = s_3w_3q_3q_4u_4t_4s_4w_4v_4p_4p_3t_3s_3.$$

Proposition 1. *D is a semiextension of C in G.*

Proof. First we note that those edges of G that belong to precisely one of C and D induce a third circuit

$$F = s_3t_3u_3q_3w_3v_3p_3p_4t_4u_4r_4v_4w_4s_4Qs_3.$$

Now it is easily checked that q_4 is the unique element of $(V(C) \cup V(D)) \setminus V(F)$. Hence if P is any path in $G - (E(C) \cup E(D))$ with both endvertices x, y in $V(C) \cup V(D)$ and no interior vertices in $V(C) \cup V(D)$, then there exists a path in F from x to y, unless $q_4 \in \{x, y\}$, in which case P is a path of length 1 with ends q_4 and w_4 (see Fig. 2). Thus D satisfies the condition for a semiextension of C, as neither of q_4, w_4 belongs to $V(C) \setminus V(D)$. ■

In [5] one additional stable circuit in G and two infinite classes of snarks with stable circuits are described. The second stable circuit in G arises as the image of C under a particular automorphism α of G. It is straightforward to check that also the circuit $\alpha(C)$ has a semiextension in G, namely the image $\alpha(D)$ of the semiextension D of C from above.

We will now turn our attention to the two infinite families from [5] of snarks with stable circuits.

Kochol in Theorem 2 of [5] constructed an infinite family of snarks each having a stable dominating circuit. This was done by recursively replacing isomorphic copies of the induced subgraphs G_1 and G_2 of G by larger graphs, each of which contains further induced subgraphs isomorphic to G_1 and/or G_2. These snarks contain stable circuits each of which intersects the subgraph induced by the vertices of $G_3 \cup G_4$ in exactly the same path that forms the intersection of C with the corresponding subgraph of G. As a consequence, the semiextension D of C in G serves equally well as a semiextension for such a circuit.

A circuit is called *dominating* in G if every edge of G is incident to at least one vertex of C. It is explained in [2] how a significant conjecture of G. Sabidussi is related to the problem of constructing a circuit double cover with a prescribed dominating circuit. In [3] we pointed out that the restriction of the semiextension conjecture to dominating circuits C implies the conjecture of Sabidussi. So far in this note we have been dealing only with semiextensions of circuits that are dominating.

Remark 3 in [5] describes a construction of snarks with stable circuits that are not dominating. With the notation introduced in [5], the construction is based on a single such snark $G^{(4)}$, with a stable circuit $C^{(3)}$. Inspection of Fig. 8 of [5] shows that $G^{(4)}$ has a subgraph with an isomorphism to the subgraph induced by $V(G_3 \cup G_4)$ in G, and that modulo this isomorphism $C^{(3)}$ is of the form

$$C^{(3)} = s_3 w_3 v_3 p_3 t_3 u_3 q_3 q_4 u_4 r_4 v_4 p_4 t_4 s_4 Q' s_3,$$

where Q' is the subpath of $C^{(3)}$ whose interior vertices lie outside the isomorphic copies of G_3 and G_4. By arguing like for G, the semiextension D of C corresponds to a semiextension of $C^{(3)}$ in $G^{(4)}$. Infinite families of examples can be constructed from $G^{(4)}$ using dot products (introduced in [4]) of $G^{(4)}$ with other snarks. The dot products may be formed in a way that does not involve edges incident to vertices of $C^{(3)}$, and so that $C^{(3)}$ remains a stable circuit in each of the snarks constructed. The semiextension for $C^{(3)}$ in $G^{(4)}$ remains a semiextension for $C^{(3)}$ in each such new example.

3. Final remarks

Every semiextension D of a circuit C that we have described in this note has the property that the symmetric difference of $E(C)$ and $E(D)$ comprises a single circuit, whereas one would in general expect the symmetric difference to form a disjoint union of several circuits. We have not been able to construct any example of a circuit which does not allow a semiextension with this property. Hence we raise the following question, which is equivalent (as one may prove straightforwardly) to asking whether this is always possible, at least in any 3-edge-connected graph.

Given sets $A, B, X \subseteq V(G)$, we say that X *separates* A and B in G if every path in G with one endvertex in A and one endvertex in B contains an element of X.

Question. If C is a circuit in a 3-edge-connected graph G, does G then contain circuits D_1, D_2 such that

(i) $E(C)$ is the symmetric difference of $E(D_1)$ and $E(D_2)$, and
(ii) $V(D_1) \cap V(D_2)$ separates $V(D_1)$ from $V(D_2)$ in G?

Here D_1 and D_2 each can play the role of the semiextension D (the other corresponding to the symmetric difference of $E(D)$ and $E(C)$).

Acknowledgments

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